MATH2230B HW3 Solution

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① $$\frac{dw}{dz} = \lim_{h\to 0} \frac{(z+h)^2 - z^2}{h}$$
$$= \lim_{h\to 0} \frac{2zh + h^2}{h}$$
$$= \lim_{h\to 0} (2z+h)$$
$$= 2z$$

② ⓐ $f' = 6z - 2$

ⓑ $f' = 5(2z^2+i)^4(8z) = 40z(2z^2+i)^4$

ⓒ $$f' = \frac{(2z+1)-(z-1)(2)}{(2z+1)^2} = \frac{3}{(2z+1)^2}$$

ⓓ $$f' = \frac{z^2\left(4(1+z^2)^3(2z)\right) - (1+z^2)^4(2z)}{z^4}$$
$$= \frac{(1+z^2)^3(8z^3 - 2z - 2z^3)}{z^4}$$
$$= \frac{2(1+z^2)^3(3z-1)}{z^3}$$

⑧ ⓐ $$\frac{f(z+h)-f(z)}{h} = \frac{\mathrm{Re}(h)}{h} = \begin{cases} 1 & \text{if } h \in \mathbb{R} \\ 0 & \text{if } h = iy,\ y \in \mathbb{R} \end{cases}$$

Thus, the limit does not exist. (DNE)

ⓑ $$\frac{f(z+h)-f(z)}{h} = \frac{\mathrm{Im}(h)}{h} = \begin{cases} 0 & \text{if } h \in \mathbb{R} \\ -i & \text{if } h = iy,\ y \in \mathbb{R} \end{cases}$$

Thus, the limit does not exist.

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① ⓐ $f(z) = \bar{z} = x - iy = u + iv$

$$\begin{cases} u_x = 1, & v_y = -1 \\ u_y = 0, & v_x = 0 \end{cases} \Rightarrow f' \text{ DNE}$$

ⓑ $f(z) = z - \bar{z} = 2yi = u + iv$

$$\begin{cases} u_x = 0, & v_y = 2 \\ u_y = 0, & v_x = 0 \end{cases} \Rightarrow f' \text{ DNE}$$

ⓒ $f(z) = 2x + ixy^2 = u + iv$

$$\begin{cases} u_x = 2, & v_y = 2xy \\ u_y = 0, & v_x = y^2 \end{cases} \cancel{\Rightarrow f' \text{ DNE}}$$

If $u_y = -v_x$, then $y = 0$, But $u_x \neq v_y$ if $y = 0$. Thus f' DNE.

ⓓ $f(z) = e^x e^{-iy} = e^x(\cos y - i\sin y)$

$$\begin{cases} u_x = e^x \cos y, & v_y = -e^x \cos y \\ u_y = -e^x \sin y, & v_x = -e^x \sin y \end{cases}$$

Similar to 1c, f' DNE.

② ⓐ $f = ix - y + 2 = 2 - y + ix = u + iv$

$$\begin{cases} u_x = 0 = v_y \\ u_y = -1 = -v_x \end{cases} \Rightarrow f' \text{ exists and } f' = i, \text{ then}$$

$f'' = 0$ exists everywhere clearly.

(b) $f(z) = e^{-x}(\cos y - i\sin y)$

$\begin{cases} u_x = -e^{-x}\cos y = v_y \\ u_y = -e^{-x}\sin y = -v_x \end{cases} \Rightarrow f'$ exists and $f' = -e^{-x}(\cos y - i\sin y)$

let $f' = p + qi$, then $\begin{cases} p_x = e^{-x}\cos y = q_y \\ p_y = e^{-x}\sin y = -q_x \end{cases} \Rightarrow f''$ exists

and $f'' = e^{-x}\cos y - e^{-x}\sin y\, i = f$

(c) $f(z) = z^3 = x^3 + 3x^2yi - 3xy^2 - y^3 i = x^3 - 3xy^2 + i(3x^2y - y^3)$

$\begin{cases} u_x = 3x^2 - 3y^2 = v_y \\ u_y = -6xy = -v_x \end{cases} \Rightarrow f'$ exists and $f' = (3x^2 - 3y^2) + 6xyi$

let $f' = p + qi$, then $\begin{cases} p_x = 6x = q_y \\ p_y = -6y = -q_x \end{cases} \Rightarrow f''$ exists.

and $f'' = 6x + 6yi = 6z$

(d) $\cancel{f(z)}$ $\begin{cases} u_x = -\sin x\cosh y = v_y \\ u_y = \cos x \sinh y = -v_x \end{cases} \Rightarrow f'$ exists and

$f' = -\sin x\cosh y - i\cos x\sinh y = p + qi$

$\begin{cases} p_x = -\cos x\cosh y = q_y \\ p_y = -\sin x\sinh y = -q_x \end{cases} \Rightarrow f''$ exists and

$f'' = -\cos x\cosh y + i\sin x\sinh y = -f$.

3a $f(z) = \frac{1}{z} = \frac{x - iy}{x^2+y^2}$

$u_x = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$, $v_y = -\left(\frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}\right)$

$= \frac{y^2 - x^2}{(x^2+y^2)^2}$ $\qquad = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\Rightarrow u_x = v_y$ if $z \neq 0$.

$u_y = \frac{\cancel{(x^2+y^2)} - x(2y)}{(x^2+y^2)^2}$ $\qquad v_x = \frac{+y(2x)}{(x^2+y^2)^2}$

$\Rightarrow u_y = -v_x$ if $z \neq 0$.

$f'(z) = -1/z^2$.

b $\begin{cases} u_x = 2x, & v_y = 2y \\ u_y = 0, & v_x = 0. \end{cases}$ $\Rightarrow f'$ exists at $x=y$, $f' = 2x$

c $f(z) = z\,\mathrm{Im}(z) = xy + iy^2$

$\begin{cases} u_x = y, & v_y = 2y \\ u_y = x, & v_x = 0 \end{cases}$ $\Rightarrow f'$ exists only at $z=0$

and $f'(0) = \lim_{z\to 0} \frac{z\,\mathrm{Im}(z)}{z} = 0$

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1a $\begin{cases} u_x = 3, & v_y = 3 \\ u_y = 1 & v_x = -1 \end{cases}$ $\Rightarrow$ f is entire.

(1b) $\begin{cases} u_x = \sinh x \cos y, & v_y = \sinh x \cos y \\ u_y = -\cosh x \sin y, & v_x = \cosh x \sin y \end{cases} \Rightarrow f$ is entire.

(1c) $\begin{cases} u_x = e^{-y} \cos x, & v_y = e^{-y} \cos x \\ u_y = -e^{-y} \sin x & v_x = e^{-y} \sin x \end{cases} \Rightarrow f$ is entire

(1d) $f = (x^2 - y^2 + 2xyi - 2)\, e^{-x} (\cos y - i \sin y)$

$= e^{-x}\Big((x^2 - y^2 - 2)\cos y + 2xy \sin y + i\big(2xy\cos y - (x^2-y^2-2)\sin y\big)\Big)$

$u_x = -e^{-x}\big((x^2-y^2-2)\cos y + 2xy \sin y\big) + e^{-x}\big(2x \cos y + 2y \sin y\big)$

$= e^{-x}\big((-x^2 + 2x + y^2 + 2)\cos y + 2y \sin y (-x+1)\big)$

$v_y = e^{-x}\big(2x(\cos y - y \sin y) - (x^2-y^2-2)\cos y + 2y \sin y\big)$

$= e^{-x}\big((-x^2+2x+y^2+2)\cos y + 2y \sin y (1-x)\big)$

$u_y = e^{-x}\big(-(x^2-y^2-2)\sin y - 2y\cos y + 2x \sin y + 2xy \cos y\big)$

$= e^{-x}\big((2x - x^2 + y^2 + 2)\sin y + (2xy - 2y)\cos y\big)$

$= \cancel{u_x} - v_x \Rightarrow f$ is entire.

(2a) $\begin{cases} u_x = y, & v_y = 1 \\ u_y = x, & v_x = 0 \end{cases} \Rightarrow$ It's diff only at $z = i$,

Thus it is nowhere analytic.

(2b) $\begin{cases} u_x = 2y, & v_y = -2y \\ u_y = 2x, & v_x = 2x \end{cases} \Rightarrow$ It is diff only at $z = 0$

It is nowhere analytic.

(2c) $f(z) = e^y(\cos x + i\sin x)$

$$\begin{cases} u_x = -e^y \sin x, & v_y = e^y \sin x \\ u_y = e^y \cos x, & v_x = e^y \cos x \end{cases}$$

It's nowhere diff. hence nowhere analytic.

(7) If f is real-valued, then $v=0$ and $u_x = u_y = 0$.
Since $u \in C^1(D)$, thus $u = $ constant.

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(3) $f = e^{\bar{z}} = e^{x-iy} = e^x(\cos y - i\sin y)$

$$\begin{cases} u_x = e^x \cos y, & v_y = -e^x \cos y \\ u_y = -e^x \sin y, & v_x = -e^x \sin y \end{cases} \Rightarrow f \text{ is nowhere analytic.}$$

(13) $U_x = e^u u_x \cos v - e^u \sin v \, v_x$

$$U_{xx} = e^u u_{xx} \cos v + e^u u_x^2 \cos v - e^u u_x v_x \sin v - e^u u_x v_x \sin v - e^u \cos v \, v_x^2 - e^u \sin v \, v_{xx}$$

Similar, $U_{yy} = e^u u_{yy} \cos v + e^u u_y^2 \cos v - e^u u_y v_y \sin v$

$$- e^u u_y v_y \sin v - e^u \cos v \, v_y^2 - e^u \sin v \, v_{yy}$$

Since $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$

$$U_{xx} + U_{yy} = e^u \cos v (u_x^2 + u_y^2 - v_x^2 - v_y^2) - 2e^u \sin v (u_x v_x - u_y v_y)$$

$$= 0$$

Similar, $V_{xx} + V_{yy} = 0$.

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① Refer to midterm 1 solution (Q3).

⑮ $\sin z = \sin x \cosh y + i \cos x \sinh y = \cosh 4$

$$\begin{cases} \sin x \cosh y = \cosh 4 \\ \cos x \sinh y = 0 \end{cases}$$

If $\cos x \sinh y = 0$, then $x = (\pi/2 + \pi n)$ or $y = 0$, $n \in \mathbb{N}$

If $y = 0$, $\sin x = \cosh 4 > 1$. There is no real root.

If $x = \pi/2 + \pi n$, $(-1)^n \cosh y = \cosh 4$

$\Rightarrow y \pm 4$ and n is even.

$\Rightarrow z = (\pi/2 + 2n\pi) \pm 4i$

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① $\sinh z = \dfrac{e^z - e^{-z}}{2} = \dfrac{1}{2}\left((e^x - e^{-x})\cos y + i(e^x + e^{-x})\sin y\right)$

$\dfrac{d}{dz}\sinh z = u_x + i v_x = \dfrac{1}{2}\left((e^x + e^{-x})\cos y + i(e^x - e^{-x})\sin y\right)$

$= \cosh z$

Similar for $\cosh z$.

~~16~~ 17 $\cos z = \cos x \cosh y - i \sin x \sinh y = 2$

$$\Rightarrow \begin{cases} \cos x \cosh y = 2 \\ \sin x \sinh y = 0 \end{cases}$$

If $\sin x \sinh y = 0$, then $x = n\pi$ or $y = 0$.

If $y = 0$, $\cos x = 2$ has no real root.

If $x = n\pi$, $(-1)^n \cosh y = 2$ has no real root if n is odd.

We consider when $x = 2n\pi$ and

$\cosh y = 2$

$e^y + e^{-y} = 4$

$(e^y)^2 - 4e^y + 1 = 0$

$$e^y = \frac{4 \pm \sqrt{16-4}}{2}$$

$= 2 + \sqrt{3}$ or $2 - \sqrt{3}$ (rej)

Thus $z = x + iy = 2n\pi \pm \log(2+\sqrt{3})$.

16a $\sinh z = \sinh x \cos y + i \cosh x \sin y = i$

$$\Rightarrow \begin{cases} \sinh x \cos y = 0 \\ \cosh x \sin y = 1 \end{cases}$$

If $\sinh x \cos y = 0$, then $x = 0$ or $y = (\pi/2 + n\pi)$ $n \in \mathbb{Z}$

If $x = 0$, $y = \frac{\pi}{2} + 2n\pi$. If $y = (\pi/2 + n\pi)$, $x = 0$ for n is even

$\Rightarrow z = \left(2n + \frac{1}{2}\right)\pi$. $n \in \mathbb{Z}$. 16b is similar to 16a.